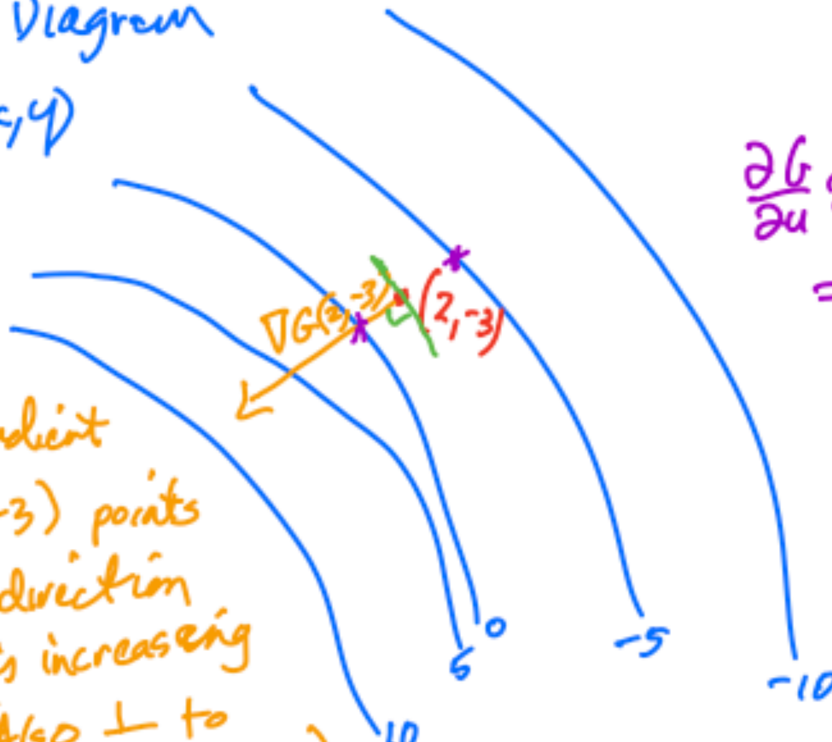
Contour Diagram
for $G(x,y)$



$$\frac{\partial G}{\partial u} \approx \frac{\Delta G}{\text{dist.}} = \frac{0-(-5)}{\text{dist}} = \frac{5}{\text{dist.}}$$

The gradient $\nabla G(2,-3)$ points in the direction where G is increasing fastest. (Also $\perp$ to contour through that point)

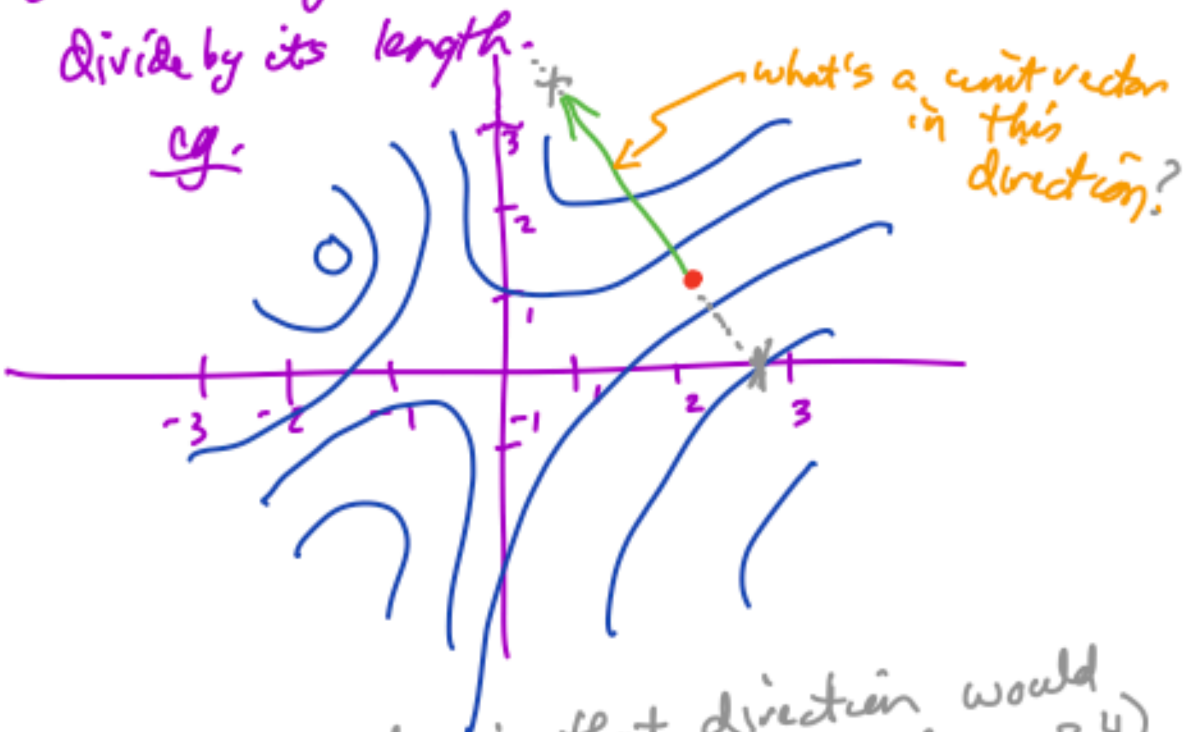
$\Longleftrightarrow$ If $u = \frac{1}{\|\nabla G(2,-3)\|}\nabla G(2,-3)$, u is the unit vector which gives the greatest directional derivative $\frac{\partial G}{\partial u}$.

Note: $$\frac{\partial G}{\partial u} = \nabla G \cdot u = \nabla G \cdot \left(\frac{1}{\|\nabla G\|}\nabla G\right)$$
$$= \frac{1}{\|\nabla G\|}\nabla G \cdot \nabla G = \frac{1}{\|\nabla G\|}\|\nabla G\|^2$$
$$= \|\nabla G\|.$$

So the length of ∇G is the directional deriv of G in that direction (unit vector).

Separate note: If you want to find a unit vector in a certain direction, just find any vector in that direction, then divide by its length.

e.g.



one vector in that direction would be $(1,3.4) - (2.7,0) \approx (-1.7, 3.4)$

A unit vector in this direction would be

$$\frac{1}{\sqrt{(1.7)^2 + (3.4)^2}}(-1.7, 3.4) = (\_\_, \_\_)$$

From before: the Hessian matrix,

or 2^{nd} derivative of a function $F(x,y,\dots)$

is the matrix

$n \times n$ square matrix.

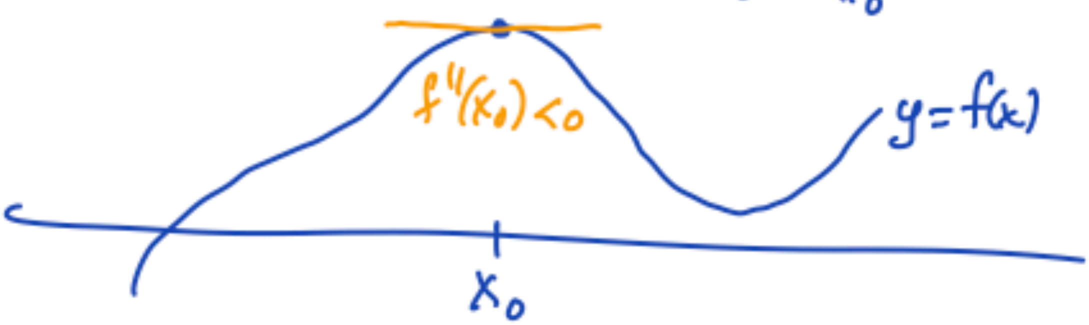
$$\begin{pmatrix} F_{xx} & F_{xy} & F_{xz} & \cdots \\ F_{yx} & F_{yy} & F_{yz} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

What does this tell us?

with fcns of 1 variable $f: \mathbb{R} \to \mathbb{R}$

$f'(x_0) = 0 \Rightarrow$ critical point.

horizontal tangent line to its graph $y = f(x)$ at x_0



3 cases:

① $f''(x_0) > 0$ positive

concave up

type: local min

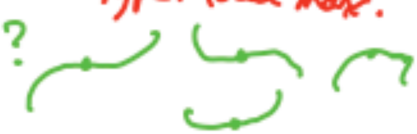
② $f''(x_0) < 0$ neg.

concave down

type: local max.

③ $f''(x_0) = 0$

?

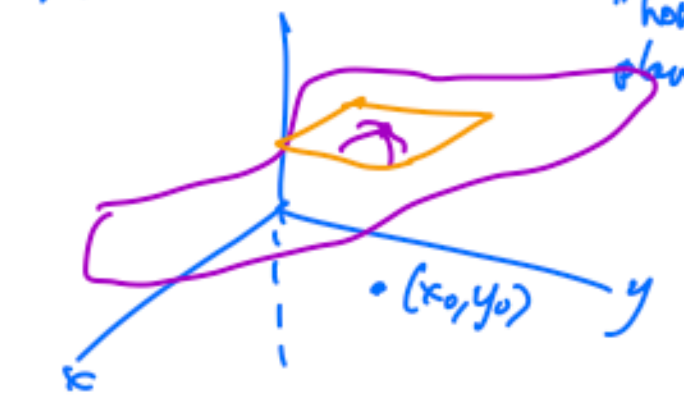


In functions of several variables:

$f : \mathbb{R}^2 \to \mathbb{R}$

$(0,0) = 0 = f' = \nabla f \rightsquigarrow$ critical points (x_0, y_0)

"horizontal tangent plane" if you graph $z = f(x,y)$.



We can tell the type of critical point by looking at the Hessian matrix at that point (x_0,y_0).

$\rightarrow$ Calculate the eigenvalues.

Each eigenvalue will tell the shape restricted to one dimension.

If $\lambda > 0 \rightarrow$ concave up

If $\lambda < 0 \rightarrow$ concave down.

If $\lambda = 0 \rightarrow$???

For 2 variables: λ_1, λ_2

both positive $\rightarrow$ local min

both negative $\rightarrow$ local max

Some one positive, other negative

Saddle point.

In higher dimensions —

all positive eigenvalues

$\Rightarrow$ local min (relative min)

all negative eigenvalues

$\Rightarrow$ local max

some $+$, some $-$ $\Rightarrow$ saddle point.

<u>Back to previous example:</u>

$$h(x,y) = \left(\tfrac{1}{2} - x^2 + y^2\right) e^{1-x^2-y^2}$$

$$\nabla h = \left(-2x\left(\tfrac{3}{2} - x^2 + y^2\right)e^{1-x^2-y^2},\ 2y\left(\tfrac{1}{2} + x^2 - y^2\right)e^{1-x^2-y^2}\right)$$

We solved for $\nabla h = (0,0)$ to get the critical points:

$$(0,0),\ \left(0,\tfrac{1}{\sqrt{2}}\right),\ \left(0,-\tfrac{1}{\sqrt{2}}\right),\ \left(\sqrt{\tfrac{3}{2}},0\right),\ \left(-\sqrt{\tfrac{3}{2}},0\right).$$

To find the type of each of these, we need to calculate the Hessian matrix.

We have $h_x = -2x\left(\frac{3}{2} - x^2 + y^2\right)e^{1-x^2-y^2}$

$$h_y = 2y\left(\frac{1}{2} + x^2 - y^2\right)e^{1-x^2-y^2}$$

Then

$$\begin{aligned} h_{xx} &= -2\left(\frac{3}{2} - x^2 + y^2\right)e^{1-x^2-y^2} \\ &\quad + (-2x)(-2x)e^{1-x^2-y^2} \\ &\quad + (-2x)\left(\frac{3}{2} - x^2 + y^2\right)e^{1-x^2-y^2} \cdot (-2x) \end{aligned}$$

$((fg)h)' = f'gh + fg'h + fgh'$